Reciprocal Lattice to sc Lattice

Simple Cubic

The primitive translation vectors of a <u>simple cubic lattice</u> may be taken as the set

$$\mathbf{a}_1 = a\hat{\mathbf{x}}$$
; $\mathbf{a}_2 = a\hat{\mathbf{y}}$; $\mathbf{a}_3 = a\hat{\mathbf{z}}$. (27a)

Here $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ are orthogonal vectors of unit length. The volume of the cell is $\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3 = a^3$. The primitive translation vectors of the reciprocal lattice are found from the standard prescription (13):

$$\mathbf{b}_1 = (2\pi/a)\hat{\mathbf{x}}$$
; $\mathbf{b}_2 = (2\pi/a)\hat{\mathbf{y}}$; $\mathbf{b}_3 = (2\pi/a)\hat{\mathbf{z}}$. (27b)

Here the reciprocal lattice is itself a simple cubic lattice, now of lattice constant $2\pi/a$.

The boundaries of the first Brillouin zones are the planes normal to the six reciprocal lattice vectors $\pm \mathbf{b}_1$, $\pm \mathbf{b}_2$, $\pm \mathbf{b}_3$ at their midpoints:

$$\pm \frac{1}{2} \mathbf{b}_1 = \pm (\pi/a) \hat{\mathbf{x}} \; ; \qquad \pm \frac{1}{2} \mathbf{b}_2 = \pm (\pi/a) \hat{\mathbf{y}} \; ; \qquad \pm \frac{1}{2} \mathbf{b}_3 = \pm (\pi/a) \hat{\mathbf{z}} \; . \tag{28}$$

The six planes bound a cube of edge $2\pi/a$ and of volume $(2\pi/a)^3$; this cube is the first Brillouin zone of the sc crystal lattice.

Reciprocal Lattice to bcc Lattice

The primitive translation vectors of the bcc lattice (Fig. 12) are

$$\mathbf{a}_{1} = \frac{1}{2}a(-\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}) ; \quad \mathbf{a}_{2} = \frac{1}{2}a(\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}) ; \quad \mathbf{a}_{3} = \frac{1}{2}a(\hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}}) , \quad (29)$$

where *a* is the side of the conventional cube and $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ are orthogonal unit vectors parallel to the cube edges. The volume of the primitive cell is

$$V = |\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3| = \frac{1}{2} a^3 . \tag{30}$$



Figure 12 Primitive basis vectors of the body-centered cubic lattice.

The primitive translations of the reciprocal lattice are defined by (13). We have, using (28),

$$\mathbf{b}_1 = (2\pi/a)(\hat{\mathbf{y}} + \hat{\mathbf{z}}) ; \qquad \mathbf{b}_2 = (2\pi/a)(\hat{\mathbf{x}} + \hat{\mathbf{z}}) ; \qquad \mathbf{b}_3 = (2\pi/a)(\hat{\mathbf{x}} + \hat{\mathbf{y}}) .$$
(31)

Note by comparison with Fig. 14 (p. 37) <u>that these are just the primitive</u> vectors of an fcc lattice, so that an fcc lattice is the reciprocal lattice of the bcc lattice.

The general reciprocal lattice vector is, for integral v_1 , v_2 , v_3 ,

$$\mathbf{G} = v_1 \mathbf{b}_1 + v_2 \mathbf{b}_2 + v_3 \mathbf{b}_3 = (2\pi/a) [(v_2 + v_3)\hat{\mathbf{x}} + (v_1 + v_3)\hat{\mathbf{y}} + (v_1 + v_2)\hat{\mathbf{z}}] \quad . \quad (32)$$

The shortest G's are the following 12 vectors, where all choices of sign are independent:

$$(2\pi/a)(\pm \hat{\mathbf{y}} \pm \hat{\mathbf{z}})$$
; $(2\pi/a)(\pm \hat{\mathbf{x}} \pm \hat{\mathbf{z}})$; $(2\pi/a)(\pm \hat{\mathbf{x}} \pm \hat{\mathbf{y}})$. (33)

One primitive cell of the reciprocal lattice is the parallelepiped described by the \mathbf{b}_1 , \mathbf{b}_2 , \mathbf{b}_3 defined by (31). The volume of this cell in reciprocal space is $\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3 = 2(2\pi/a)^3$. The cell contains one reciprocal lattice point, because each of the eight corner points is shared among eight parallelepipeds. Each parallelepiped contains one-eighth of each of eight corner points. In solid state physics we take the central (Wigner-Seitz) cell of the reciprocal lattice as the first Brillouin zone. Each such cell contains one lattice point at the central point of the cell. <u>This zone (for the bcc lattice) is bounded by the</u> <u>planes normal to the 12 vectors of Eq. (32) at their midpoints.</u> The zone is a regular 12-faced solid, a rhombic dodecahedron, as shown in Fig. 13. The <u>vectors from the origin to the center of each face are</u>

$$(\pi/a)(\pm \hat{\mathbf{y}} \pm \hat{\mathbf{z}}) ; \qquad (\pi/a)(\pm \hat{\mathbf{x}} \pm \hat{\mathbf{z}}) ; \qquad (\pi/a)(\pm \hat{\mathbf{x}} \pm \hat{\mathbf{y}}) . \tag{33}$$

All choices of sign are independent, giving 12 vectors.





The long diagonal of each face is exactly $\sqrt{2}$ times the length of the short diagonal, so that the acute angles on each face measure arc Cos(1/3), or approximately 70.53°

Reciprocal Lattice to the fcc Lattice



Figure 14 Primitive basis vectors of the face-centered cubic lattice.

Reciprocal Lattice to fcc Lattice

The primitive translation vectors of the fcc lattice of Fig. 14 are

$$\mathbf{a}_1 = \frac{1}{2}a(\hat{\mathbf{y}} + \hat{\mathbf{z}}) \; ; \; \mathbf{a}_2 = \frac{1}{2}a(\hat{\mathbf{x}} + \hat{\mathbf{z}}) \; ; \; \mathbf{a}_3 = \frac{1}{2}a(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \; .$$
 (34)

The volume of the primitive cell is

$$V = |\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3| = \frac{1}{4} a^3 \quad . \tag{35}$$

The primitive translation vectors of the lattice reciprocal to the fcc lattice are

$$\mathbf{b}_1 = (2\pi/a)(-\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}) ; \qquad \mathbf{b}_2 = (2\pi/a)(\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}) ;$$

$$\mathbf{b}_3 = (2\pi/a)(\hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}}) .$$
(36)

These are primitive translation vectors of a bcc lattice, so that the bcc lattice is reciprocal to the fcc lattice. The volume of the primitive cell of the reciprocal lattice is $4(2\pi/a)^3$.

The shortest **G**'s are the eight vectors:

$$(2\pi/a)(\pm \hat{\mathbf{x}} \pm \hat{\mathbf{y}} \pm \hat{\mathbf{z}}) \quad . \tag{37}$$

The boundaries of the central cell in the reciprocal lattice are determined for the most part by the eight planes normal to these vectors at their midpoints. But the corners of the octahedron thus formed are cut by the planes that are the perpendicular bisectors of six other reciprocal lattice vectors:

$$(2\pi/a)(\pm 2\hat{\mathbf{x}})$$
; $(2\pi/a)(\pm 2\hat{\mathbf{y}})$; $(2\pi/a)(\pm 2\hat{\mathbf{z}})$. (38)

Note that $(2\pi/a)(2\hat{\mathbf{x}})$ is a reciprocal lattice vector because it is equal to $\mathbf{b}_2 + \mathbf{b}_3$. The first Brillouin zone is the smallest bounded volume about the origin, the truncated octahedron shown in Fig. 15. The six planes bound a cube of edge $4\pi/a$ and (before truncation) of volume $(4\pi/a)^3$. for a bcc lattice in a real space $\mathbf{a}_2 + \mathbf{a}_3 = 2 \ \mathbf{a}_1$ For a bcc lattice in reciprocal lattice $\mathbf{b}_2 + \mathbf{b}_3 = 2 \ \mathbf{b}_3$





 $4\pi/a$

Truncated Octahedron

Figure 15 Brillouin zones of the face-centered cubic lattice. The cells are in reciprocal space, and the reciprocal lattice is body centered.

FOURIER ANALYSIS OF THE BASIS

When the diffraction condition $\Delta \mathbf{k} = \mathbf{G}$ of Eq. (21) is satisfied, the scattering amplitude (18) for a crystal of N cells may be written as

$$F_{\mathbf{G}} = N \int_{cell} dV \, n(\mathbf{r}) \, \exp(-i\mathbf{G} \cdot \mathbf{r}) = NS_{\mathbf{G}} \quad . \tag{39}$$

The quantity $S_{\mathbf{G}}$ is called the **structure factor** and is defined as an integral over a single cell, with $\mathbf{r} = 0$ at one corner.

Often it is useful to write the electron concentration $n(\mathbf{r})$ as the superposition of electron concentration functions n_j associated with each atom jof the cell. If \mathbf{r}_j is the vector to the center of atom j, then the function $n_j(\mathbf{r} - \mathbf{r}_j)$ defines the contribution of that atom to the electron concentration at \mathbf{r} . The total electron concentration at \mathbf{r} due to all atoms in the single cell is the sum

$$n(\mathbf{r}) = \sum_{j=1}^{s} n_j (\mathbf{r} - \mathbf{r}_j)$$
(40)

over the *s* atoms of the basis. The decomposition of $n(\mathbf{r})$ is not unique, for we cannot always say how much charge density is associated with each atom. This is not an important difficulty.

The structure factor defined by (39) may now be written as integrals over the *s* atoms of a cell:

$$S_{\mathbf{G}} = \sum_{j} \int d\mathbf{V} \, n_{j}(\mathbf{r} - \mathbf{r}_{j}) \, \exp(-i\mathbf{G} \cdot \mathbf{r})$$

= $\sum_{j} \exp(-i\mathbf{G} \cdot \mathbf{r}_{j}) \int d\mathbf{V} \, n_{j}(\boldsymbol{\rho}) \, \exp(-i\mathbf{G} \cdot \boldsymbol{\rho}) \, ,$ ⁽⁴¹⁾

where $\rho = \mathbf{r} - \mathbf{r}_{j}$. We now define the **atomic form factor** as

$$f_j = \int dV \, n_j(\boldsymbol{\rho}) \, \exp(-i\mathbf{G} \cdot \boldsymbol{\rho}) \, , \qquad (42)$$

integrated over all space. If $n_i(\boldsymbol{\rho})$ is an atomic property, f_i is an atomic property.

We combine (41) and (42) to obtain the structure factor of the basis in the form

$$S_{\mathbf{G}} = \sum_{j} f_{j} \exp(-i\mathbf{G} \cdot \mathbf{r}_{j}) \quad . \tag{43}$$

The usual form of this result follows on writing for atom *j*:

$$\mathbf{r}_j = x_j \mathbf{a}_1 + y_j \mathbf{a}_2 + z_j \mathbf{a}_3 \quad , \tag{44}$$

as in (1.2). Then, for the reflection labelled by v_1 , v_2 , v_3 , we have

$$\mathbf{G} \cdot \mathbf{r}_{j} = (v_{1}\mathbf{b}_{1} + v_{2}\mathbf{b}_{2} + v_{3}\mathbf{b}_{3}) \cdot (x_{j}\mathbf{a}_{1} + y_{j}\mathbf{a}_{2} + z_{j}\mathbf{a}_{3}) = 2\pi(v_{1}x_{j} + v_{2}y_{j} + v_{3}z_{j}) , \qquad (45)$$

so that (43) becomes

$$S_{\mathbf{G}}(v_1 v_2 v_3) = \sum_j f_j \exp[-i2\pi (v_1 x_j + v_2 y_j + v_3 z_j)] \quad .$$
(46)

The structure factor S need not be real because the scattered intensity will involve S^*S , where S^* is the complex conjugate of S so that S^*S is real.

Structural Factor of the bcc Lattice

The bcc basis referred to the cubic cell has identical atoms at $x_1 = y_1 = z_1 = 0$ and at $x_2 = y_2 = z_2 = \frac{1}{2}$. Thus (46) becomes

$$S(v_1 v_2 v_3) = f\{1 + \exp[-i\pi(v_1 + v_2 + v_3)]\}, \qquad (47)$$

where f is the form factor of an atom. The value of S is zero whenever the exponential has the value -1, which is whenever the argument is $-i\pi \times (\text{odd integer})$. Thus we have

$$\begin{array}{ll} S=0 & \text{when } v_1+v_2+v_3=\text{ odd integer }; \\ S=2f & \text{when } v_1+v_2+v_3=\text{ even integer }. \end{array}$$

(47a)

Metallic sodium has a bcc structure. The diffraction pattern does not contain lines such as (100), (300), (111), or (221), but lines such as (200), (110), and (222) will be present; here the indices $(v_1v_2v_3)$ are referred to a cubic cell.



Figure 16 Explanation of the absence of a (100) reflection from a body-centered cubic lattice. The phase difference between successive planes is π , so that the reflected amplitude from two adjacent planes is $1 + e^{-i\pi} = 1 - 1 = 0$.

Structural Factor of the fcc Lattice

The basis of the fcc structure referred to the cubic cell has identical atoms at 000; $0\frac{11}{22}$; $\frac{1}{2}0\frac{1}{2}$; $\frac{11}{22}0$. Thus (46) becomes

$$S(v_1 v_2 v_3) = f[1 + \exp[-i\pi(v_2 + v_3)] + \exp[-i\pi(v_1 + v_3)] + \exp[-i\pi(v_1 + v_2)]]$$

$$(48)$$

If all indices are even integers, S = 4f; similarly if all indices are odd integers. But if only one of the integers is even, two of the exponents will be odd multiples of $-i\pi$ and S will vanish. If only one of the integers is odd, the same argument applies and S will also vanish. Thus in the fcc lattice no reflections can occur for which the indices are partly even and partly odd.





Figure 17 We may construct the sodium chloride crystal structure by arranging Na⁺ and Cl⁻ ions alternately at the lattice points of a simple cubic lattice. In the crystal each ion is surrounded by six nearest neighbors of the opposite charge. The space lattice is fcc, and the basis has one Cl⁻ ion at 000 and one Na⁺ ion at $\frac{111}{222}$. The figure shows one conventional cubic cell. The ionic diameters here are reduced in relation to the cell in order to clarify the spatial arrangement.



K⁺ is at (0,0,0), and Cl⁻ is at (1/2, 1/2, 1/2).

as if is for a simple cubic of a/2 lattice constant. According to eq. 47a, only even integers in the indices will give reflection intensities

Figure 17 Comparison of x-ray reflections from KCl and KBr powders. In KCl the numbers of electrons of K^+ and Cl^- ions are equal. The scattering amplitudes $f(K^+)$ and $f(Cl^-)$ are almost exactly equal, so that the crystal looks to x-rays as if it were a monatomic simple cubic lattice of lattice constant a/2. Only even integers occur in the reflection indices when these are based on a cubic lattice of lattice constant a. In KBr the form factor of Br⁻ is quite different to that of K⁺, and all reflections of the fcc lattice are present. (Courtesy of R. van Nordstrand.)

Atomic Form Factor

In the expression (46) for the structure factor, there occurs the quantity f_j , which is a measure of the scattering power of the *j*th atom in the unit cell. The value of f involves the number and distribution of atomic electrons, and the wavelength and angle of scattering of the radiation. We now give a classical calculation of the scattering factor.

The scattered radiation from a single atom takes account of interference effects within the atom. We defined the form factor in (42):

$$f_j = \int dV \, n_j(\mathbf{r}) \, \exp(-i\mathbf{G} \cdot \mathbf{r}) \quad , \tag{49}$$

with the integral extended over the electron concentration associated with a single atom. Let \mathbf{r} make an angle α with \mathbf{G} ; then $\mathbf{G} \cdot \mathbf{r} = Gr \cos \alpha$. If the electron distribution is spherically symmetric about the origin, then

$$\begin{split} f_j &\equiv 2\pi \int dr \, r^2 \, d(\cos \alpha) \, n_j(r) \, \exp(-iGr \cos \alpha) \\ &= 2\pi \int dr \, r^2 n_j(r) \cdot \, \frac{e^{iGr} - e^{-iGr}}{iGr} \, , \end{split}$$

after integration over $d(\cos \alpha)$ between -1 and 1. Thus the form factor is given by

$$f_j = 4\pi \int dr \, n_j(r) r^2 = \frac{\sin Gr}{Gr} \quad . \tag{50}$$

If the same total electron density were concentrated at r = 0, only Gr = 0 would contribute to the integrand. In this limit $(\sin Gr)/Gr = 1$, and

$$f_j = 4\pi \int dr \, n_j(r) r^2 = Z \quad , \tag{51}$$

the number of atomic electrons. Therefore f is the ratio of the radiation amplitude scattered by the actual electron distribution in an atom to that scattered by one electron localized at a point. In the forward direction G = 0, and freduces again to the value Z.

The overall electron distribution in a solid as seen in x-ray diffraction is fairly close to that of the appropriate free atoms. This statement does not mean that the outermost or valence electrons are not redistributed somewhat in forming the solid; it means only that the x-ray reflection intensities are represented well by the free atom values of the form factors and are not very sensitive to small redistributions of the electrons.

Homework problem set of Chapter 2

- 1. Problem No. 5.
- 2. Problem No. 6.
- 3. Together with the problem set of Chapter 1, It is due on the Monday, Oct. 5, class time.